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Generalized Schmidt number for multipartite states

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The Schmidt number is of crucial importance in characterizing the bipartite pure states. We explore and propose here the definition of Schmidt number for states in multipartite systems. It is shown to be entanglement monotonic and valid for both pure and mixed states. Our approach is applicable for systems with arbitrary number of parties and for arbitrary dimensions.

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I. INTRODUCTION

Entanglement is shown to play a crucial role in quantum information processing and quantum computation [1]. However, quantifying entanglement is not so straightforward, and is becoming one of the most significant problems in this area [2, 3]. Several kinds of entanglement measures have been proposed for bipartite case [2–4]. Yet there are no operational methods for multipartite states in general.

In contrast to the bipartite case, the situation is more involved in the multipartite case. There are many kinds of entanglement. For the simplest case, a three-qubit state can be either fully separable, biseparable, or genuinely entangled. An m-partite state might have many different kinds of cases: fully separable, 2-separable, 3-separable, …, $(m-2)$-separable, and genuinely entangled, etc. On the other hand, the structure of the dimensionality of the multipartite case is a intricate one. The bipartite pure state $|\psi\rangle$ is uniquely determined by its reductions but a tripartite pure state has three single-particle marginals of inequivalent rank. It is generally difficult to characterize different types of multipartite entanglement and distinguish them from each other completely.

The Schmidt decomposition [5] is indispensable in the characterization and quantification of entanglement associated with pure states [2–4, 6–9]. The *Schmidt number* (or Schmidt rank sometimes) can be used to characterize and quantify the degree of bipartite entanglement for pure state directly [4, 6] and, also, it can be operationally interpreted as the zero-error entanglement cost in the protocol of one-shot entanglement dilution [7]. However, the Schmidt decomposition is not valid for multipartite case. Only rare pure states in the multipartite case admit the generalized Schmidt decomposition $|\psi\rangle=\sum_{k=1}^{\min\{N_1,N_2,\dots,N_m\}}|e_k^{(1)}\rangle|e_k^{(2)}\rangle\otimes\cdots\otimes|e_k^{(m)}\rangle$ [10, 11], where N_i denotes the dimension of the $i-$th subsystem. For the simplest three-partite case, any three-qubit pure state can be written as $|\psi\rangle=\lambda_0|000\rangle+\lambda_1e^{i\theta}|100\rangle+\lambda_2|101\rangle+\lambda_3|110\rangle+\lambda_4|111\rangle$, where $\lambda_i\geq 0$, $\sum_i\lambda_i^2=1$, $\theta\in[0,\pi]$ [12], which is not a form of Schmidt decomposition. That is, there is no correspondence of Schmidt number for multipartite case. In this paper, first we try to extend the Schmidt number to multipartite systems.

Let $|\psi\rangle=\sum_{i,j}d_{ij}|i_1\rangle|j_2\rangle$ be a pure state lives in $\mathbb{C}^{N_1}\otimes\mathbb{C}^{N_2}$ and $|\psi\rangle=\sum_k\lambda_k|e_k\rangle|f_k\rangle$ be its Schmidt decomposition, where $\{|i_1\rangle\}$ and $\{|j_2\rangle\}$ are computational bases of $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$, respectively. Then the Schmidt number of $|\psi\rangle$ is defined by

$$R_\psi=\mathrm{rank}(\rho_1)=\mathrm{rank}(\rho_2),$$

where ρ_i denotes the reduced state of the $i-$th part. λ_ks are called the Schmidt coefficient of $|\psi\rangle$. It is clear that R_ψ coincides with the rank of the coefficient matrix of $|\psi\rangle$, i.e., $R_\psi=\mathrm{rank}(D)$, $D=[d_{ij}]$; R_ψ is also the length of the Schmidt decomposition of $|\psi\rangle$. For mixed state ρ it is defined by [19]

$$R_\rho=\inf_{\mathcal{D}(\rho)}\max_{\psi_i}R_{\psi_i},$$

where $\mathcal{D}(\rho)=\{p_i,|\psi_i\rangle:\rho=\sum_ip_i|\psi_i\rangle\langle\psi_i|\}$ is for all pure ensembles of ρ. Almost any entanglement measure for pure states can be represented by its Schmidt coefficient [4]. The von Neumann entanglement entropy of bipartite pure state $|\psi\rangle$ is

$$E(|\psi\rangle)=S(\rho_1)=S(\rho_2)=-\sum_k\lambda_k^2\log_2\lambda_k^2,$$

where $S(\cdot)$ denotes the von Neumann entropy. The entanglement of formation for mixed states ρ is defined as

$$E_f(\rho)=\inf_{\mathcal{D}(\rho)}\sum_kp_kE(|\psi_k\rangle),$$

where $\inf\limits_{\mathcal{D}(\rho)}$ denotes the infimum over all pure ensembles of ρ.

II. THE SCHMIDT NUMBER FOR MULTIPARTITE CASE

We develop a method of extending the Schmidt number to multipartite case. We consider the three qubit case first. If $|\psi\rangle$ is fully separable (i.e., 1|2|3 separable), then $|\psi\rangle=|\psi_1\rangle|\psi_2\rangle|\psi_3\rangle$. In such a case, the Schmidt number of $|\psi\rangle$ can be defined to be 1. Suppose $|\psi\rangle$ is bi-separable, without loss of generality, we assume that it is 1|23 separable and not fully separable, i.e., $|\psi\rangle=|\psi_1\rangle|\psi_{23}\rangle$. Since $|\psi_{23}\rangle$ is a bipartite entangled pure state, it has Schmidt number $R_{\psi_{23}}=2$, and

TABLE I. The Schmidt number of the three qubit pure state (r_i denotes the rank of ρ_i, '$i-j$' means ρ_{ij} is not a pure separable state, '$i\quad j-k$' means $\rho_{jk}=|\psi_{jk}\rangle\langle\psi_{jk}|$ is an entangled pure state, etc.)

Type	Model	Reductions	R_ψ
1|2|3	1 2 3	$r_i=1$	1
1|23	1 $2-3$	$r_1=1$, $r_2=r_3=2$	2
12|3	$1-2$ 3	$r_3=1$, $r_1=r_2=2$	2
2|13	2 $1-3$	$r_2=1$, $r_1=r_3=2$	2
GE	figure (a)	$r_i=2$, $\rho_{\bar{i}}$ is separable, $i=1,2,3$	3
GE	figure (a)	$r_i=2$, $\rho_{\bar{i}}$ is entangled for some i	4

then R_ψ can be viewed as $R_{\psi_{23}}$, namely, $R_\psi=2$. If $|\psi\rangle$ is genuinely entangled, then $\mathrm{rank}(\rho_1)=\mathrm{rank}(\rho_{23})=\mathrm{rank}(\rho_2)=\mathrm{rank}(\rho_{13})=\mathrm{rank}(\rho_3)=\mathrm{rank}(\rho_{12})=2$, where ρ_γ denotes the reduction of the $\gamma-$subsystem(s). If ρ_{12}, ρ_{23} and ρ_{13} are separable, then $R_{\rho_{12}}=R_{\rho_{13}}=R_{\rho_{23}}=1$, we thus view R_ψ as $\mathrm{rank}(\rho_1)+R_{\rho_{23}}=\mathrm{rank}(\rho_2)+R_{\rho_{13}}=\mathrm{rank}(\rho_3)+R_{\rho_{12}}=3$. If ρ_{23}, or ρ_{12}, or ρ_{13} is entangled, then $\max\{R_{\rho_{12}},R_{\rho_{23}},R_{\rho_{13}}\}=2$, we thus view R_ψ as $\mathrm{rank}(\rho_1)+2=4$. That is, there are four types of entanglement indeed for the three qubit case (see Table I). (For $i\in\{1,2,3,\dots,m\}$, we denote by $\bar{i}$ the combination consisting of all elements in $\{1,2,\dots,m\}-\{i\}$, for instance, if $m=4$, $i=(2)$, then $\bar{i}=(134)$.)

We now move to the $N_1\otimes N_2\otimes N_3$ case. We may assume that $N_1\le N_2\le N_3$. If $|\psi\rangle$ is fully separable, then $|\psi\rangle=|\psi_1\rangle|\psi_2\rangle|\psi_3\rangle$ and thus the Schmidt number of $|\psi\rangle$ can be considered to be 1. If $|\psi\rangle$ is bi-separable, without loss of generality, we assume that it is 1|23 separable and not fully separable, i.e., $|\psi\rangle=|\psi_1\rangle|\psi_{23}\rangle$. Since $|\psi_{23}\rangle$ is a bipartite entangled pure state, we let $R_{\psi_{23}}=t$, $2\le t\le N_2$, then R_ψ can be viewed as $R_{\psi_{23}}$, that is, $R_\psi=t$. If $|\psi\rangle$ is genuinely entangled, then $\mathrm{rank}(\rho_1)=\mathrm{rank}(\rho_{23})=i\ge 2$, $\mathrm{rank}(\rho_2)=\mathrm{rank}(\rho_{13})=j\ge 2$ and $\mathrm{rank}(\rho_3)=\mathrm{rank}(\rho_{12})=k\ge 2$. If ρ_{12}, ρ_{23} and ρ_{13} are separable, then $R_{\rho_{12}}=R_{\rho_{13}}=R_{\rho_{23}}=1$, we thus view R_ψ as $\max_i(\mathrm{rank}(\rho_i)+R_{\rho_{\bar{i}}})\ge 3$. If ρ_{23}, or ρ_{12}, or ρ_{13} is entangled, then $\max\{R_{\rho_{12}},R_{\rho_{23}},R_{\rho_{13}}\}\ge 2$, we thus view R_ψ as $\max_i(\mathrm{rank}(\rho_i)+R_{\rho_{\bar{i}}})\ge 4$. That is, there are at most N_1+N_3 types of entanglement for the three qubit case (see Table II).

Note.—If $m>3$, then there exist $|\psi\rangle$ and $|\phi\rangle$ in $m-$partite systems, such that $|\psi\rangle$ is $k-$separable, $|\phi\rangle$ is genuinely entangled but $R_\psi>R_\phi$.

A natural way of generalizing the Schmidt number to mixed states ρ acting on $\mathbb{C}^{N_1}\otimes\mathbb{C}^{N_2}\otimes\mathbb{C}^{N_3}$ can be defined by the convex roof structure

$$R_\rho:=\inf_{\mathcal{D}(\rho)}\max_{\psi_i}R_{\psi_i},\tag{1}$$

where the minimization is over all ensemble of ρ. It is reasonable as this means that ρ cannot be obtained by mixing pure states with Schmidt number lower than R_ρ and that there exists an ensemble with Schmidt number at most R_ρ to reach the state. It is clear that R_ρ is an entanglement monotone since local rank of pure state is non-increasing under local operations and classical communication (LOCC) [20].

TABLE II. The Schmidt number of $|\psi\rangle$ in $N_1\otimes N_2\otimes N_3$ system.

Type	Reductions	R_ψ
1|2|3	$r_i=1$	1
1|23	$r_1=1$, $r_2=r_3=R_{\psi_{23}}$	$R_{\psi_{23}}$
12|3	$r_3=1$, $r_1=r_2=R_{\psi_{12}}$	$R_{\psi_{12}}$
2|13	$r_2=1$, $r_1=r_3=R_{\psi_{13}}$	$R_{\psi_{13}}$
GE	$r_i\ge 2$, $i=1,2,3$	$\max_i(r_i+R_{\rho_{\bar{i}}})$

TABLE III. The Schmidt number of $|\psi\rangle$ in $N_1\otimes N_2\otimes N_3\otimes N_4$ system.

Type	Model	Local rank	R_ψ
1|2|3|4	1 2 3 4	$r_i=1$	1
1|23|4	1 $2-3$ 4	$r_1=r_4=1$	$R_{\psi_{23}}$
12|3|4	$1-2$ 3 4	$r_3=r_4=1$	$R_{\psi_{12}}$
2|3|14	2 3 $1-4$	$r_2=r_3=1$	$R_{\psi_{14}}$
2|13|4	2 $1-3$ 4	$r_2=r_4=1$	$R_{\psi_{13}}$
1|3|24	1 3 $2-4$	$r_1=r_3=1$	$R_{\psi_{24}}$
1|2|34	1 2 $3-4$	$r_1=r_2=1$	$R_{\psi_{34}}$
1|234	1 $2-3-4$	$r_1=1$	$R_{\psi_{234}}$
2|134	2 $1-3-4$	$r_2=1$	$R_{\psi_{134}}$
3|124	3 $1-2-4$	$r_3=1$	$R_{\psi_{124}}$
4|123	4 $1-2-3$	$r_4=1$	$R_{\psi_{123}}$
12|34	$1-2$ $3-4$	$r_i\ge 2$	$R_{\psi_{12}}+R_{\psi_{34}}$
13|24	$1-3$ $2-4$	$r_i\ge 2$	$R_{\psi_{13}}+R_{\psi_{24}}$
14|23	$1-4$ $2-3$	$r_i\ge 2$	$R_{\psi_{14}}+R_{\psi_{23}}$
GE	figure (b)	$r_i\ge 2$	$\max_i(r_i+R_{\rho_{\bar{i}}})$

Now we can establish the Schmidt number for the four-partite system as Table III. Analogously, we can define the Schmidt number for mixed states via the convex roof structure as Eq. (1). This approach can be extended to $m-$partite case step by step for both pure and mixed states. That is, for $|\psi\rangle\in\mathbb{C}^{N_1}\otimes\mathbb{C}^{N_2}\otimes\cdots\otimes\mathbb{C}^{N_m}$, the R_ψ can be defined as the program discussed above; for mixed state ρ acting on $\mathbb{C}^{N_1}\otimes\mathbb{C}^{N_2}\otimes\cdots\otimes\mathbb{C}^{N_m}$, it can be defined by the convex roof structure

$$R_\rho:=\inf_{\mathcal{D}(\rho)}\max_{\psi_i}R_{\psi_i},\tag{2}$$

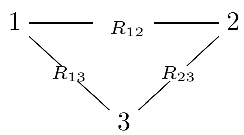
where the minimization is over all ensemble of ρ. We now can conclude the following result.

Theorem. The Schmidt number defined as Eq. (2) is an entanglement monotone.

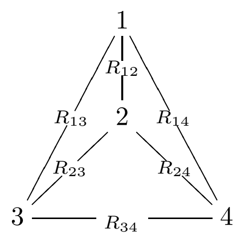
Similar to the bipartite case, a multipartite ρ is fully separable iff $R_\rho=1$. We have now established a complete hierarchy of Schmidt numbers that quantify the dimensions of the entanglement. The bipartite Schmidt number can be viewed as a universal entanglement measure since the Schmidt number fully reflect the dimensional

of entanglement[6]. From this point of view, the generalized Schmidt number can also be viewed as a universal entanglement measure for the multipartite case.

We illustrate the Schmidt number for genuinely entanglement with the following figures. '$i - R_{ij} - j$' means ρ_{ij} is a mixed state with Schmidt number R_{ij}. For the tripartite and the four-partite case, the Schmidt number



(a)



(b)

can be explained by figures (a) and (b) respectively. In (b), R_ψ is determined by r_i and $R_{\bar{i}}$ while $R_{\bar{i}}$ is decided by the Schmidt number of the six bipartite reductions and the four single part reductions.

Note.— It is straightforward that the generalized Schmidt number is invariant under the invertible SLOCC (stochastic local operations and classical communication) since invertible SLOCC preserves the rank of the reduction [21] (also see in [22–24]).

III. THE SCHMIDT COEFFICIENT FOR MULTIPARTITE CASE

Based on the scenario of defining multipartite Schmidt number, we can extend the Schmidt coefficient to multipartite systems. We begin with the tripartite case. Let $|\psi\rangle$ be a pure state in a $N_1 \otimes N_2 \otimes N_3$ system. If it is fully separable, the Schmidt coefficient is $\{1\}$. If it is $1|23$ separable, the Schmidt coefficient is defined as that of $|\psi_{23}\rangle$; similarly, we can define it for the types $12|3$ and $2|13$. If it is genuinely entangled, we assume without loss of generality that

$$\begin{aligned} R_\psi &= \mathrm{rank}(\rho_{t_1}) + R_{\rho_{\bar{t_1}}} \\ &= \mathrm{rank}(\rho_{t_2}) + R_{\rho_{\bar{t_2}}} = \cdots = \mathrm{rank}(\rho_{t_k}) + R_{\rho_{\bar{t_k}}}. \end{aligned}$$

Let $\tilde{\rho}_{t_i} = \frac{1}{\sqrt{2}}\rho_{t_i}^{\frac{1}{2}}$, and let

$$\sigma(\tilde{\rho}_{t_i}) = \{\lambda_p^{(i)}\}_{p=1}^{\mathrm{rank}(\tilde{\rho}_{t_i})},$$

where $\sigma(\cdot)$ denotes the set of eigenvalues of the described matrix. Let $|\psi_1^{(t_i)}\rangle$, $|\psi_2^{(t_i)}\rangle$, $\ldots$, $|\psi_r^{(t_i)}\rangle$ be elements in the pure state ensembles of $\rho_{\bar{t_i}}$ such that $R_{\psi_j^{(t_i)}} = R_{\rho_{\bar{t_i}}}$, $j = 1, 2, \ldots, r$. Assume that

$$E(|\psi_{j_0}^{(t_i)}\rangle) = \max_j E(|\psi_j^{(t_i)}\rangle),$$

$\mathcal{S}_C(|\psi_{j_0}^{(i)}\rangle) = \{\lambda_q^{(i)}\}$ and $\delta_q^{(i)} = \frac{\lambda_q^{(i)}}{\sqrt{2}}$, where $\mathcal{S}_C(x)$ denotes the Schmidt coefficient of x. We write

$$E_i(|\psi\rangle) := \begin{cases} -\sum\limits_p (\lambda_p^{(i)})^2 \log_2(\lambda_p^{(i)})^2 - \sum\limits_q (\delta_q^{(i)})^2 \log_2(\delta_q^{(i)})^2 & \text{if } R_{\rho_{\bar{t_i}}} > 1, \\ -\sum\limits_p (\lambda_p^{(i)})^2 \log_2(\lambda_p^{(i)})^2 + \frac{1}{2} & \text{if } R_{\rho_{\bar{t_i}}} = 1. \end{cases}$$

If E_i reaches a maximum for some i with $R_{\rho_{\bar{t_i}}} = 1$,

$$\sigma(\tilde{\rho}_{t_i}) \bigcup \{\frac{1}{\sqrt{2}}\} \tag{3}$$

is defined to be the Schmidt coefficient of $|\psi\rangle$; If E_i reaches a maximum for some i with $R_{\rho_{\bar{t_i}}} > 1$, denote $\mathcal{S}_C(|\psi_{j_0}^{(i)}\rangle)$ by $\mathcal{S}_C^{t_i}$,

$$\sigma(\tilde{\rho}_{t_i}) \bigcup \mathcal{S}_C^{t_i} \tag{4}$$

is defined to be the Schmidt coefficient of $|\psi\rangle$. The ratio '$\frac{1}{\sqrt{2}}$' here guarantees that the Schmidt coefficient is normalized, i.e., the sum of the squares is 1.

For the four-partite system, if it is not genuinely entangled, then it reduces to the three-partite case. For example, if $|\psi\rangle$ is $1|234$ separable, then $|\psi\rangle = |\psi_1\rangle|\psi_{234}\rangle$. So we can define the Schmidt coefficient as that of $|\psi_{234}\rangle$. If $|\psi\rangle$ is $12|34$ separable, then $|\psi\rangle = |\psi_{12}\rangle|\psi_{34}\rangle$. In such a case, we define the Schmidt coefficient to be $\sigma(\tilde{\rho}_1)\bigcup\sigma(\tilde{\rho}_3)$. In these cases, the von Neumann entanglement entropy is clear. If it is genuinely entangled, we let $|\phi_0^{(i)}\rangle$ be an element in the pure state ensembles of $\rho_{\bar{i}}$ such that $R_\psi = \mathrm{rank}(\rho_i) + R_{\rho_{\bar{i}}}$, $R_{\rho_{\bar{i}}} = R_{\phi_0^{(i)}}$ and $E(|\phi_0^{(i)}\rangle)$ reaches the maximal over all elements $|\phi^{(i)}\rangle$s in the ensembles of $\rho_{\bar{i}}$ such that $R_{\rho_{\bar{i}}} = R_{\phi^{(i)}}$, where $E(|\phi_0^{(i)}\rangle)$ is defined as $E(|\psi_0^{(i)}\rangle) := \max\limits_j E_j(|\psi_0^{(i)}\rangle)$, $1 \le j \le 4$. If

$$S(\tilde{\rho}_{i_0}) + E(|\phi_0^{(i_0)}\rangle) = \max_i\{S(\tilde{\rho}_i) + E(|\phi_0^{(i)}\rangle)\}, \tag{5}$$

we let $\mathcal{S}_C(|\phi_0^{(i_0)}\rangle) = \{\gamma_k\}$ and $\tilde{\mathcal{S}}_C^{i_0} = \{\tilde{\gamma}_k\}$ with $\tilde{\gamma}_k = \frac{\gamma_k}{\sqrt{2}}$. Then we call

$$\sigma(\tilde{\rho}_{i_0}) \bigcup \tilde{\mathcal{S}}_C^{i_0} \tag{6}$$

the Schmidt coefficient of $|\psi\rangle$.

Note.—(i) ρ is fully separable iff the Schmidt coefficient is $\{1\}$; (ii) the number of the Schmidt coefficient coincides with the Schmidt number; (iii) the Schmidt coefficient may not be unique.

IV. EXAMPLES

We end our discussion with some examples. Two well known three qubit states are the W state and the GHZ state,

$$|W_3\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle),$$
$$|\mathrm{GHZ}_3\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$

For the state $|W\rangle$, one can easily see that any coefficient matrix of any bipartite splitting is not of rank-one, so it is genuinely entangled. From Table I, $R_{W_3} = 4$. The state $|\mathrm{GHZ}_3\rangle$ is also genuinely entangled. Table I indicates $R_{\mathrm{GHZ}_3} = 3$. In [21], it is proved that $|W_3\rangle$ and $|\mathrm{GHZ}_3\rangle$ are two types of genuinely entangled states under SLOCC classification, which meets our results. The Schmidt coefficient of $|W_3\rangle$ is $\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{2}, \frac{1}{2}\}$. The Schmidt coefficient of $|\mathrm{GHZ}_3\rangle$ is $\{\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\}$. In addition, the entanglement dimensionality vector [25] of $|W_3\rangle$, (2, 2, 2), coincides with that of $|\mathrm{GHZ}_3\rangle$. From this point of view the generalized Schmidt number provides a more strict classification of multipartite states than the scenario of entanglement dimensionality vector proposed in [25]. In addition, it is worth noticing that the Schmidt number is different from the *collectibility* proposed in [26] since the collectibility of $|\mathrm{GHZ}_3\rangle$ is larger than that of $|W_3\rangle$.

For the $m-$qubit W-state $|W_m\rangle$ and the GHZ state

$$|W_m\rangle = \frac{1}{\sqrt{m}}(|0\cdots01\rangle + |0\cdots010\rangle + |1\cdots00\rangle),$$
$$|\mathrm{GHZ}_m\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes m} + |1\rangle^{\otimes m}),$$

one can check that $R_{W_m} = 2(m-1)$ and $R_{\mathrm{GHZ}_m} = 3$. It is worth mentioning here that all the reductions of the $|W_m\rangle$ are genuinely entangled while all the reductions of the $|\mathrm{GHZ}_m\rangle$ are (fully) separable. Similarly, for the generalized GHZ state in the $d^{\otimes m}$ system, the Schmidt number of $|\mathrm{GHZ}_m^{(d)}\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}(|i\rangle^{\otimes m})$ is $d+1$.

V. CONCLUSION

In this letter, the Schmidt number and the Schmidt coefficient for multipartite case are established for the first time. We showed that the Schmidt number is a well-defined entanglement measure since it is entanglement monotonic. Our results may shed new lights on the task of multipartite systems: the multipartite states can be classified via the generalized Schmidt number.

Going further, one can define the generalized entanglement formation in terms of the Schmidt coefficient. That is, if the Schmidt coefficient of $|\psi\rangle$ is $\{\eta_i\}$, then we can define the generalized entanglement of formation by $E(|\psi\rangle) := -\sum_i \eta_i^2 \log_2 \eta_i^2$. It can be extended to mixed states via the convex roof structure. (Note that although the Schmidt coefficient may not be unique the generalized von Neumann entanglement entropy is unique.) The origin entanglement of formation for the bipartite case is an entanglement monotone, we conjecture that the generalized entanglement of formation is an entanglement monotone (the proof maybe a hard work due to the complex structure of both the multipartite states and the multipartite LOCC).

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